Sum of n squares

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Prove that:

$$\sum_{n=1}^{m} n^2 = \frac{n(n+1)(2n-1)}{6}$$

Basis of induction:

- We are able to show that n(1) is true since:

$$\frac{1(1+1)(2-1)}{6} = 1^2 = 1$$

- Through induction, we want to prove that this equation is true for all values k, where $k \geq 1$. This means that k+1 is also true.

Proof by Induction:

- Let that the following is true is for any number of n:

$$\sum_{n=1}^{m} n^2$$

- We now have to prove that n+1 is also true in this equation:

$$\sum_{n=1}^{m+1} n^2$$

- Using the properties of summation, we get:

$$\sum_{n=1}^{m+1} n^2 = \sum_{n=1}^{m} n^2 + (n+1)^2$$

- Following hypothesis of our proof, we can substitute the summation equation for $\frac{n(n+1)(2n-1)}{6}$:

$$\sum_{n=1}^{m+1} n^2 = \frac{n(n+1)(2n-1)}{6} + (n+1)^2$$

- Expand the right side:

$$= \frac{n(n+1)(2n-1)}{6} + (n+1)^2$$

$$= \frac{n(2n^2+n-1)}{6} + \frac{6(n^2+2n+1)}{6}$$

$$= \frac{2n^3+9p^2+13p+6}{6}$$

- Since we are using induction to solve this equation, the original n in the front of the equation is now $n+1$. With this this, we can use synthetic divison to find the factors:

$$\begin{array}{r|rrrr} -1 & 2 & 9 & 13 & 6 \\ \hline & & -2 & -7 & -6 \\ \hline & 2 & 7 & 6 & 0 \end{array}$$

- Through even more simplification, this is the simplified equation:

$$\frac{(n+1)(p+2)(2p+3)}{6}$$

- Analyzing the equation, we are able to see that n is substituted by n+1 in our equation:

$$\frac{(k+1)((k+1)+1)(2(k+1)-1)}{6}$$

- Therefore, since both n and n+1 are true, $\frac{n(n+1)(2n-1)}{6}$ is true for $n \in Z$